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NUMERICAL OUTFLOW BOUNDARY CONDITIONS FOR THE SHALLOW-WATER EQUATIONS

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SUMMARY

A simple explanation is given of the occurrence of wiggles in the flow field near outflow boundaries. If the shallow-water equations are solved numerically spurious solutions with an oscillatory character turn out to exist, which can be generated by certain additional numerical boundary conditions on the downstream side. The wiggles usually damp quickly with the distance from the boundary. Some ways of handling the downstream boundary are given which largely avoid the occurrence of wiggles.

KEY WORDS Shallow-Water Equations Spurious Solutions Numerical Boundary Condition Advection

PROBLEM

In numerical solutions of the shallow-water equations, it is often observed that 'wiggles' occur in the flow pattern near an outflow boundary. This is sometimes attributed to non-linear effects, but a simple explanation is lacking. In the present note, a very simplified linear analysis is given which explains the principle of the wiggles.

BOUNDARY CONDITIONS

From the theory of partial differential equations, it is known that the shallow-water equations (with viscosity terms disregarded) require two boundary conditions on an inflow boundary and one at outflow.[2,3] The latter and one of the inflow conditions will usually be a specified water level or normal velocity. The additional inflow condition, physically related to vorticity transport, may involve the tangential velocity component. In many numerical methods, special approximations are used near the boundaries, which may be interpreted as additional boundary conditions.

SCHEMATIZATION

A number of simplifications can be made. As the problems occur both in steady and unsteady flow, time dependence is not essential. Also it is observed that the wiggles in the velocity field are not accompanied by water-level variations, so that a constant depth can be assumed for the purpose. Bottom friction is not essential but it will be taken into account. The remaing equations are then

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} + c_f\bar{u}(\bar{u}^2 + \bar{v}^2)^{1/2} = 0 \tag{1}$$

$$\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y} + c_f\bar{v}(\bar{u}^2 + \bar{v}^2)^{1/2} = 0 \tag{2}$$

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where x and y are the horizontal co-ordinates, $\bar{u}$ and $\bar{v}$ the corresponding depth-averaged velocities and c_f a bottom-friction factor. For simplicity, assume that the region is a rectangle, contained within $0 \leqslant x \leqslant L$ and that the outflow boundary is at $x = L$. According to the previous section, a boundary condition for $\bar{u}$ can be imposed either at $x = 0$ or $x = L$; a condition for $\bar{v}$ is acceptable only at $x = 0$.

To further simplify the analysis, let us linearize the equations by defining

$$(\bar{u}, \bar{v}) = (U, V) + (u, v)$$

where u, v are supposed to be small. The linearized equations then become

$$U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y} + \lambda_1 u = 0 \tag{3}$$

$$U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y} + \lambda_2 u = 0 \tag{4}$$

with

$$\lambda_1 = c_f(U^2 + V^2)^{1/2}\left(1 + \frac{U^2}{U^2 + V^2}\right)$$

and

$$\lambda_2 = c_f(U^2 + V^2)^{1/2}\left(1 + \frac{V^2}{U^2 + V^2}\right)$$

Mixed terms due to the friction term have been neglected for simplicity, so that the two equations are now uncoupled. Their analytical solutions are

$$u(x, y) = u_0\left(y - \frac{V}{U}x\right)\exp\left(-\frac{\lambda_1}{U}x\right) \tag{5}$$

and similarly for v. The boundary conditions at upstream or downstream sides (for u) and upstream (for v) determine u_0 and v_0.

NUMERICAL SOLUTION

Using central differences, equation (3) is approximated as

$$\frac{U}{2\Delta x}(u_{k+1,j} - u_{k-1,j}) + \frac{V}{2\Delta y}(u_{k,j+1} - u_{k,j-1}) + \lambda u_{k,j} = 0 \tag{6}$$

and similarly for equation (4). The solution of equation (6) contains contributions of the form

$$u_{k,j} = u_1 r^k s^j \tag{7}$$

where r and s satisfy (for $\Delta x = \Delta y$)

$$(r - r^{-1}) + V/U(s - s^{-1}) + 2\mu = 0 \tag{8}$$

with $\mu = \lambda\Delta x/U$. For each s, there are two possible values for r, so that the solution reads

$$u_{k,j} = (u_1 r_1^k + u_2 r_2^k)s^j \tag{9}$$

in which u_1 and u_2 are fixed by the boundary conditions. Now assume that a boundary condition is applied on the upstream side, such that a slow variation in the y direction is present, i.e. $s = \exp(i\eta)$

with $\eta = K\Delta y$ small. Assuming μ to be small too, we find approximately

$$r_1 = 1 - \mu - \mathrm{i}\varepsilon, \quad r_2 = -1 - \mu - \mathrm{i}\varepsilon \tag{10}$$

with $\varepsilon = V/U \sin\eta$. Now, the first part of this corresponds to the required solution (5):

$$r_1^k s^j = (1 - \mu - \mathrm{i}\varepsilon)^k \mathrm{e}^{\mathrm{i}j\eta} \approx \exp\left\{ -\frac{\lambda}{U}x + \mathrm{i}\eta\left(j - \frac{v}{U}k\right)\right\}$$

The second solution r_2 is parasitic and it has an oscillatory character, with an amplitude increase in the positive x direction. The amplification factor of the amplitude per grid interval exceeds $(1 + \mu)$.

TREATMENT OF OUTFLOW BOUNDARY

Some typical ways of handling the outflow boundary $x_n = L$ will be discussed. Other methods can be analysed the same way. It is noted that the behaviour of u and v at the outflow boundary is completely similar; therefore, only the u component is discussed.

(i) $(\partial u/\partial x) = 0$ approximated by $u_{n,j} = u_{n-1,j}$. Together with a boundary condition on the inflow side, this gives two conditions for u_1 and u_2:

$$u_1 + u_2 = u_0 \tag{11}$$

$$u_1 r_1^n + u_2 r_2^n = u_1 r_1^{n-1} + u_2 r_2^{n-1} \tag{12}$$

or

$$u_2 = -u_1(r_1^n - r_1^{n-1})/(r_2^n - r_2^{n-1})$$

This gives approximately

$$u_2 \approx (-1)^n \tfrac{1}{2}(\mu + \mathrm{i}\varepsilon)u_1 \tag{13}$$

which will be small of first order $O(\Delta x, \Delta y)$. The component with u_1 will then approximately give the correct solution.

(ii) $(\partial u/\partial x) = 0$ approximated by $u_{n+1,j} = u_{n-1,j}$, where $n + 1$ is a virtual point outside the region. In this case equation (12) will be replaced by

$$u_1 r_1^{n+1} + u_2 r_2^{n+1} = u_1 r_1^{n-1} + u_2 r_2^{n-1} \tag{14}$$

and consequently

$$u_2 = -u_1(r_1^{n+1} - r_1^{n-1})/(r_2^{n+1} - r_2^{n-1}) \approx (-1)^n \frac{\mu - \mathrm{i}\varepsilon}{\mu + \mathrm{i}\varepsilon} u_1 \tag{15}$$

The parasitic solution turns out to be of the same order as the physical one and errors up to 100 per cent are obtained.

(iii) $(\partial^2 u/\partial x^2) = 0$ approximated by $u_{n+1,j} - 2u_{n,j} + u_{n-1,j} = 0$. Equation (12) is now replaced by

$$u_1(r_1^{n+1} - 2r_1^n + r_1^{n-1}) + u_2(r_2^{n+1} - 2r_2^n + r_2^{n-1}) = 0 \tag{16}$$

leading to

$$u_2 \approx (-1)^n \frac{\varepsilon^2}{4} u_1 \tag{17}$$

which is of order $O(\Delta y^2)$ and, therefore, more accurate than case (i). A similar result, not shown here, is obtained by using the equations at the boundary $x_n = L$, with one-sided differences for the x-derivatives. Again, a second-order contribution for u_2 is found, and, therefore, a second-order accurate main component u_1. This agrees with the result by Gustafsson[1] that a boundary treatment one order less accurate than internal points will not decrease the overall order of convergence.

NUMERICAL VERIFICATION

To illustrate the effect of various outflow boundary conditions, some numerical examples were generated using the DUCHESS program of the Delft University of Technology. A simple square basin of $10 \times 10\,\text{km}^2$ was taken, with a water depth of 30 m and an inflow velocity of 1 m/s. The side walls were closed. In order to have a cross-flow component at the outflow section, the water level at outflow was tilted by 0·1 m across the basin. Some further data:

bottom friction coefficient:	$c_f = 0{\cdot}004$
grid size:	$\Delta x = \Delta y = 500\,\text{m}$
time step:	$\Delta t = 180\,\text{s}$
total simulated time:	$t = 18{,}000\,\text{s}$

The standard version of the program uses method (i) at the outflow section. As a comparison, the same run was made with method (ii) of the preceding section. The results are shown in Figure 1 at time 18000 s; the flow is then about steady.

In the standard case, almost no wiggles are observed; the numerical output, however, shows some small amplitude oscillations in the v component. In the case of method (ii), very strong wiggles occur in global agreement with the theory. A quantitative comparison is not so easy to make; the growth rate of the v component is found to be about $-1{\cdot}5$ per grid interval, whereas the theory would yield $v_2 \approx -3$. However, it should be realized that the theory applies to small disturbances and the numerical examples to very strong ones. Generally speaking, the conclusions of the theoretical analysis are supported.

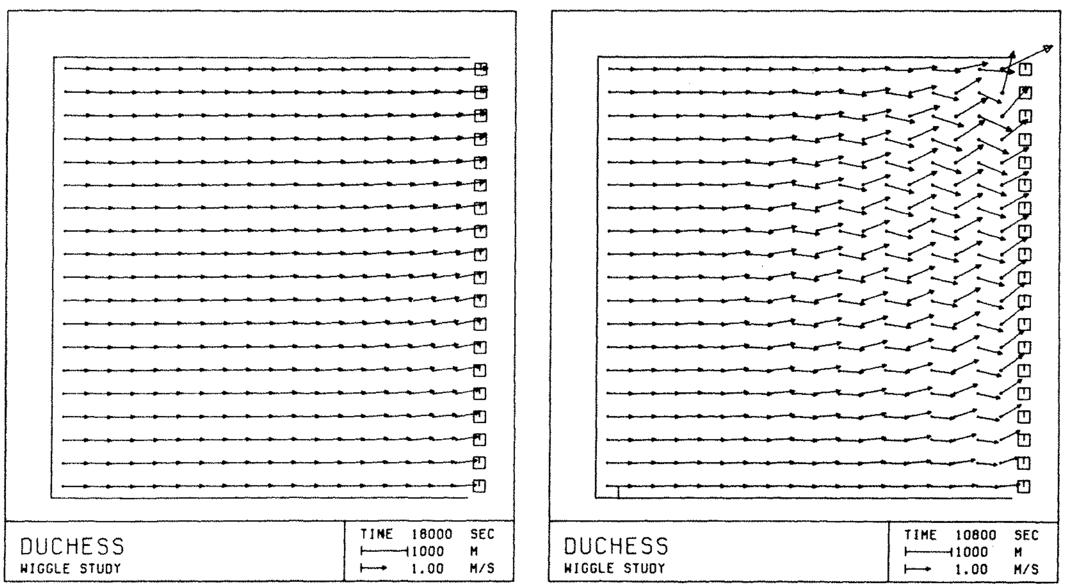

Figure 1. Computed steady-state flow pattern. Outflow boundary treated by method (i) (left) and by method (ii) (right).

DISCUSSION AND CONCLUSION

A simple explanation has been given of wiggles in the velocity field near the outflow boundary for the shallow-water equations. The occurrence of wiggles turns out to be perfectly explicable by linear theory, and is related with the specification of additional, numerical boundary conditions. Good methods to avoid the greater part of the wiggles are simple first-order extrapolation (over one mesh interval!), second-order extrapolation, or using the differential equations approximated by one-sided differences in the direction normal to the boundary. The theory applies to staggered grids as well.

In some cases, similar wiggles are observed upstream of obstacles (islands) in the flow. The present theory does not apply there, as there is no outflow at such boundaries. Therefore, such internal wiggles and ways to avoid them require a more sophisticated theory.

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